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LINES, LINE-POINT INCIDENCES AND CROSSING FAMILIES IN DENSE SETS

PAVEL VALTR*

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Let $\mathcal{P}$ be a set of n points in the plane. We say that $\mathcal{P}$ is *dense* if the ratio between the maximum and the minimum distance in $\mathcal{P}$ is of order $O(\sqrt{n})$. A set C of line segments in the plane is called *a crossing family* if the relative interiors of any two line segments of C intersect. Vertices of line segments of a crossing family C are called *vertices of* C. It is known that for any set $\mathcal{P}$ of n points in general position in the plane there is a crossing family of size $\Omega(\sqrt{n})$ with vertices in $\mathcal{P}$. In this paper we show that if $\mathcal{P}$ is dense then there is a crossing family of almost linear size with vertices in $\mathcal{P}$.

The above result is related to well-known results of Beck and of Szemerédi and Trotter. Beck proved that any set $\mathcal{P}$ of n points in the plane, not most of them on a line, determines at least $\Omega(n^2)$ different lines. Szemerédi and Trotter proved that if $\mathcal{P}$ is a set of n points and $\mathcal{L}$ is a set of m lines then there are at most $O(m^{2/3}n^{2/3}+m+n)$ incidences between points of $\mathcal{P}$ and lines of $\mathcal{L}$. We study whether or not the bounds shown by Beck and by Szemerédi and Trotter hold for any dense set $\mathcal{P}$ even if the notion of incidence is extended so that a point is considered to be incident to a line l if it lies in a small neighborhood of l. In the first case we get very close to the conjectured bound $\Omega(n^2)$. In the second case we obtain a bound of order $O(\min\{m^{3/4}n^{5/8}\log^{1/4}n, n\sqrt{m}\})$.

1. Introduction

This paper is motivated by the investigations of so-called dense sets and by the following three results:

Theorem 1. (Szemerédi–Trotter [9]) *Let $\mathcal{P}$ be a set of n points in the plane and let $\mathcal{L}$ be a family of m lines in the plane. Then the number of incidences between points in $\mathcal{P}$ and lines in $\mathcal{L}$ is at most $O(m^{2/3}n^{2/3}+m+n)$.*

Theorem 2. (Beck [5]) *Let $\mathcal{P}$ be a set of n points in the plane and let x be an integer such that no $n-x$ points of $\mathcal{P}$ lie on a line. Then the set $\mathcal{P}$ determines at least $\Omega(xn)$ lines.*

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Theorem 3. (Aronov *et al.* [2]) *Let $\mathcal{P}$ be a set of n points in general position in the plane. Then there exists a collection $\mathcal{C}$ of at least $\sqrt{n/24}$ line segments with end-points in $\mathcal{P}$ such that the relative interiors of any two segments of $\mathcal{C}$ intersect.*

Theorems 1 and 2 gave an answer to well-known problems; the bounds in these theorems are tight (up to a constant factor). On the other hand, there is no sublinear upper bound for the statement in Theorem 3.

We are interested in a strengthening of Theorems 1–3 for so-called dense sets.

For two points x and y in the plane, let $|xy|$ denote the distance between x and y. Let $\mathcal{P}$ be a set of n points in the plane. We define

$$q(\mathcal{P}) = \frac{\max\{|ab| : a, b \in \mathcal{P}\}}{\min\{|ab| : a, b \in \mathcal{P}, a \neq b\}}.$$

For $\alpha > 0$, we say that the set $\mathcal{P}$ is *α-dense,* if $q(\mathcal{P}) \leq \alpha\sqrt{n}$. It is known that one can find an arbitrarily large α-dense set if and only if $\alpha \geq \sqrt{2\sqrt{3}/\pi} \approx 1.05$. Combinatorial and computational properties of dense sets have recently been studied in a couple of works ([1, 3, 6, 10, 11, 12]).

A set $\mathcal{C}$ of line segments in the plane is called *a crossing family* if the relative interiors of any two line segments of $\mathcal{C}$ intersect. Vertices of line segments of a crossing family $\mathcal{C}$ are called *vertices of $\mathcal{C}$*. Thus, Theorem 3 shows that for any set $\mathcal{P}$ of n points in general position in the plane there is a crossing family of size $\Omega(\sqrt{n})$ with vertices in $\mathcal{P}$.

The bounds in Theorems 1 and 2 are tight and this remains true if we restrict our attention to α-dense sets with any fixed $\alpha \geq \sqrt{2\sqrt{3}/\pi}$. Therefore, we may and shall investigate whether or not Theorems 1 and 2 hold for α-dense sets even if the notion of incidence is extended so that a line is assumed to be incident to all points which lie on it or close to it. For the exact formulation we need some notation.

Let $\mathcal{P}$ be an α-dense set of size n. Assume that the minimum distance in $\mathcal{P}$ equals 1. We say that a point and a line are *roughly incident* if their distance is smaller than $1/\sqrt{n}$. We say that two lines determined by two pairs of points of $\mathcal{P}$ are *essentially different* if their directions differ at least by $1/n$ or if their $(1/\sqrt{n})$-neighborhoods do not intersect inside the convex hull of $\mathcal{P}$.

Note that Theorem 2 implies that, for a fixed $\alpha \geq \sqrt{2\sqrt{3}}/\pi$, any α-dense set of size n determines $\Omega(n^2)$ lines. We have the following two conjectures which would respectively generalize Theorems 1 and 2 in the case of dense sets.

Conjecture 4. *Assume that $\alpha \geq \sqrt{2\sqrt{3}}/\pi$ is fixed. If $\mathcal{P}$ is an α-dense set of size n and $\mathcal{L}$ is a family of m pairwise essentially different lines in the plane then the number of rough incidences between points in $\mathcal{P}$ and lines in $\mathcal{L}$ is at most $O(\min\{m\sqrt{n}, m^{2/3}n^{2/3}, n^2\})$.*

Conjecture 5. *Assume that $\alpha \geq \sqrt{2\sqrt{3}/\pi}$ is fixed. Then any α-dense set $\mathcal{P}$ of size n determines $\Omega(n^2)$ pairwise essentially different lines.*

The definitions of the notions "roughly incident" and "essentially different lines" are naturally balanced so that Conjectures 4 and 5 are not trivially false. This can be seen on the example $\mathcal{P}$ is a square grid $\sqrt{n} \times \sqrt{n}$.

Besides their clear theoretical appeal, Conjectures 4 and 5 can also be seen from the view of computational geometry. In many problems in computational geometry there is no big difference if three points lie on a line or if they lie almost on a line. A transparent and yet quite general situation is then represented by dense sets. The restriction on dense sets isn't so restrictive as it may seem at first sight since most results on dense sets are also valid for sets which contain large dense subsets.

In the case of Theorem 3 we conjecture the following strengthening for dense sets.

Conjecture 6. *For every fixed $\alpha \geq \sqrt{2\sqrt{3}/\pi}$, any α-dense set of size n contains vertices of a crossing family of size $\Omega(n)$.*

The following three theorems which are somewhat weaker than Conjectures 4–6 represent the main results of this paper.

Theorem 7. *Assume that $\alpha \geq \sqrt{2\sqrt{3}/\pi}$ is fixed. If $\mathcal{P}$ is an α-dense set of size n and $\mathcal{L}$ is a family of m pairwise essentially different lines in the plane then the number of rough incidences between points in $\mathcal{P}$ and lines in $\mathcal{L}$ is at most $O(\min\{m\sqrt{n}, m^{3/4}n^{5/8}\log^{1/4} n, n\sqrt{m}, n^2\})$.*

Theorem 8. *Assume that $\alpha \geq \sqrt{2\sqrt{3}/\pi}$ is fixed. Then any α-dense set $\mathcal{P}$ of size n determines $\Omega\left(\frac{n^2}{(\log n)^{4\log\log\log n+c}}\right)$ pairwise essentially different lines, where $c = 4\log(1000\alpha^2)+8$.*

Theorem 9. *For every fixed $\alpha \geq \sqrt{2\sqrt{3}/\pi}$, any α-dense set of size n contains vertices of a crossing family of size $\Omega\left(\frac{n}{(\log n)^{4\log\log\log n+c}}\right)$, where $c = 4\log(1000\alpha^2)+8$.*

All logarithms in Theorems 7–9 (and also everywhere else in this paper) are assumed to be of base two. Theorems 8 and 9 are close to Conjectures 5 and 6, respectively, while there is a more substantial gap between Conjecture 4 and Theorem 7.

Note that the bound $O(\min\{m\sqrt{n}, m^{3/4}n^{5/8}\log^{1/4} n, n\sqrt{m}, n^2\})$ in Theorem 7 equals

(i) $O(m\sqrt{n})$ if $m \leq \sqrt{n}\log n$,

(ii) $O(m^{3/4}n^{5/8}\log^{1/4} n)$ if $\sqrt{n}\log n \leq m \leq n^{3/2}/\log n$,

(iii) $O(n\sqrt{m})$ if $n^{3/2}/\log n \leq m \leq n^2$, and

(iv) $O(n^2)$ if $m \geq n^2$.

Theorem 7 also yields the following corollary (which is somewhat weaker but more readable than Theorem 7).

Corollary 10. *Assume that $\alpha \geq \sqrt{2\sqrt{3}}/\pi$ is fixed. If $\mathcal{P}$ is an α-dense set of size n and $\mathcal{L}$ is a family of m pairwise essentially different lines in the plane then the number of rough incidences between points in $\mathcal{P}$ and lines in $\mathcal{L}$ is at most $O(\min\{m\sqrt{n}, m^{7/10}n^{7/10}\log^{1/5} n, n^2\})$.*

Aronov *et al.* [2] also proved an algorithmic colored version of Theorem 3.

Theorem 11. (Aronov *et al.* [2]) *Let $\mathcal{P}$ be a set of n points in general position, and let $n/2$ points of $\mathcal{P}$ be colored blue and the remaining $n/2$ points of $\mathcal{P}$ be colored red. Then there exists a crossing family of size at least $\sqrt{n/24}$ whose every line segment connects a blue point with a red point. Moreover, one can find such a crossing family in time $O(n \log n)$.*

Theorems 8 and 9 have similar versions:

Theorem 12. *Let $\alpha \geq \sqrt{2\sqrt{3}}/\pi$ be fixed. Then, for any $\alpha-$dense set $\mathcal{P}$ of size n containing $n/2$ blue and $n/2$ red points, there exist $\Omega\left(\frac{n^2}{(\log n)^{4\log\log\log n+c}}\right)$ pairwise essentially different lines, each incident to at least one blue point and to at least one red point, where $c = 4\log(1000\alpha^2)+8$.*

Theorem 13. *Let $\alpha \geq \sqrt{2\sqrt{3}}/\pi$ be fixed. Then, for any $\alpha-$dense set of size n containing $n/2$ blue and $n/2$ red points, there exists a crossing family of size $\Omega\left(\frac{n}{(\log n)^{4\log\log\log n+c}}\right)$, where $c = 4\log(1000\alpha^2)+8$, whose every line segment connects a blue point with a red point. Moreover, one can find such a crossing family deterministically in time $O(n^3)$ or by a randomized algorithm in the expected time $O(n \log n)$.*

We can also apply our approach to random point sets:

Theorem 14. (i) *Let $\mathcal{P}$ be a set of n points which are randomly and independently chosen from a disk. Then, with a high probability, there is a crossing family of size $\frac{1}{10}n - O(\sqrt{n})$ with vertices in $\mathcal{P}$. Moreover, there is a deterministic algorithm which, with a high probability, finds such a crossing family in time $O(n \log n)$.*

(ii) *If $\mathcal{P}$ is a set of n points which are randomly and independently chosen from a bounded planar convex body B of positive area then, with a high probability, there is a crossing family of size $\frac{6}{100}n - O(\sqrt{n})$ with vertices in $\mathcal{P}$.*

The paper is organized as follows. In Section 2 we describe the key part of our approach and prove weaker versions of Theorems 8 and 9. In Section 3 we do a more precise calculation giving Theorems 8 and 9. Theorem 7 and Corollary 10 are proved in Section 4. In Section 5 we give the proofs of Theorems 12–14 and conclude the paper with two conjectures.

Throughout the whole paper we shall assume that $\alpha \geq \sqrt{2\sqrt{3}/\pi}$ is fixed and that $\mathcal{P}$ is an α-dense set. For our purposes, we may and shall assume that $\min\{|ab| : a,b \in \mathcal{P}\} = 1$. So $q(\mathcal{P})$ is the diameter of $\mathcal{P}$ in this case. The size of $\mathcal{P}$ shall be denoted by n. A disk of radius r with center C shall be denoted by $D(C,r)$.

The paper is partially based on a part of the author's PhD. thesis [12].

2. A crossing family of almost linear size

In this section we give the proofs of the following two theorems which give slightly worse bounds than Theorems 8 and 9.

Theorem 15. *For every fixed $\alpha > \sqrt{2\sqrt{3}/\pi}$ and $\varepsilon > 0$, any α-dense set $\mathcal{P}$ of size n determines $\Omega(n^{2-\varepsilon})$ pairwise essentially different lines.*

Theorem 16. *For every fixed $\alpha > \sqrt{2\sqrt{3}/\pi}$ and $\varepsilon > 0$, any α-dense set $\mathcal{P}$ of size n contains vertices of a crossing family of size $\Omega(n^{1-\varepsilon})$.*

The proofs of Theorems 15 and 16 use basic ideas of the proofs of Theorems 8 and 9 in a simpler form.

Throughout the whole section we assume that $n = |\mathcal{P}|$ is a large positive integer.

Let $D = D\left(C, \frac{\sqrt{n}}{8}\right)$ be a disk of radius $\frac{\sqrt{n}}{8}$, and let $m = \left\lfloor \frac{\pi}{8} n \right\rfloor$. Fix an angle $\varphi \in [0, 2\pi)$. For $i = 1, 2, \ldots, 2m$, let R_i be a rectangle with the side lengths $s = \alpha\sqrt{n}$ and $t = \frac{1}{\sqrt{n}}$ touching the disk D in the point $T_i = C + \left(\frac{\sqrt{n}}{8}\cos(\varphi + \frac{i}{2m}2\pi), \frac{\sqrt{n}}{8}\sin(\varphi + \frac{i}{2m}2\pi)\right)$ whereas T_i is the middle point of a side of R_i of length t (see Fig. 1). Note that the rectangles $R_i, i = 1, 2, \ldots, 2m$, are pairwise disjoint because the total length $2m\frac{1}{\sqrt{n}}$ of their sides touching D does not exceed the length $2\pi\frac{\sqrt{n}}{8}$ of the boundary circle of D.

An ordered pair $B = (R, R')$ of rectangles R and R' will be called *a double-rectangle* if $R \cup R'$ is congruent to $R_1 \cup R_{m+1}$. (Two sets in the plane are called *congruent* if we can rotate and translate one of them so that it will coincide with the other one.) In particular, if $B = (R, R')$ is a double-rectangle then the rectangles R and R' are of size $s \times t$ each. For any $i = 1, \ldots, m$, the pair $B_i = (R_i, R_{m+i})$ of the two opposite rectangles R_i and R_{m+i} is obviously a double-rectangle (see Fig. 1).

We say that an ordered pair $(a, a') \in \mathcal{P}^2$ of points of $\mathcal{P}$ *occupies* a double-rectangle $B = (R, R')$ if $a \in R$ and $a' \in R'$. A double-rectangle will be called *occupied* if at least one pair of points of $\mathcal{P}$ occupies it.

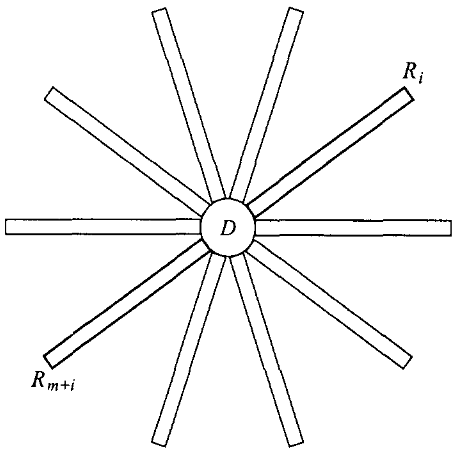

Fig. 1. Rectangles R_i and a double-rectangle $B_i = (R_i, R_{m+i})$

Observation 17. *If k of the double-rectangles $B_i = (R_i, R_{m+i}), i = 1, 2, \ldots, m$, are occupied, then there is a crossing family of size k with vertices in $\mathcal{P}$.*

Proof. For any occupied double-rectangle B_i, we take one pair of points of $\mathcal{P}$ which occupies B_i, and connect it by a line segment. We obtain a crossing family of size k since every two line segments intersect inside D. ∎

In the sequel, we will show that it is possible to place the center C of D and to find φ so that at least $\Omega(n^{1-\varepsilon})$ double-rectangles are occupied. Then Observation 17 will immediately yield Theorem 16. We will proceed by probabilistic methods, taking C and φ at random.

For a double-rectangle $B = (R, R')$, the direction of the vector which translates R' into R will be called *the direction of* B, and the center of symmetry of B will be called *the center of* B. So, for any $i = 1, 2, \ldots, m$, the center of B_i is C, and the direction of B_i is $\varphi + \frac{i}{2m} 2\pi = \varphi + \frac{i}{m}\pi$ (if $\varphi + \frac{i}{m}\pi < 2\pi$) or $\varphi + \frac{i}{m}\pi - 2\pi$ (otherwise).

Now we can define a set of double-rectangles with a natural measure on it. Fix an arbitrary point O in $\mathcal{P}$. Hence, the entire set $\mathcal{P}$ lies in the disk $D(O, s)$. Let $\mathcal{M}$ be the set of all double-rectangles $B = (R, R')$ with center in the disk $D' = D\left(O, s + \frac{t}{2}\right) = D\left(O, \alpha\sqrt{n} + \frac{1}{2\sqrt{n}}\right)$. (The radius of D' is chosen so that $\mathcal{M}$ contains all occupied double-rectangles.) Since $\mathcal{M}$ is equivalent to the product of D' and $[0, 2\pi)$ (corresponding to the placement of the center of B and to the direction of B, respectively), there is a measure μ on $\mathcal{M}$ which is equivalent to the product of Lebesgue measures on D' and on $[0, 2\pi)$. Note that the measure of the whole set $\mathcal{M}$ is $\mu(\mathcal{M}) = \pi\left(s + \frac{t}{2}\right)^2 \cdot 2\pi = \Theta(n)$.

Define

$$\mathcal{M}(x, y) = \{B \in \mathcal{M} : (x, y) \text{ occupies } B\}, \text{ for } x, y \in \mathcal{P},$$

$$\mathcal{M}_{\text{oc}} = \bigcup_{(x,y)\in\mathcal{P}^2} \mathcal{M}(x,y) = \{B \in \mathcal{M} : B \text{ is occupied}\}.$$

The following two lemmas show that a lower bound on the measure of $\mathcal{M}_{oc}$ gives both a lower bound on the maximum size of a crossing family with vertices in $\mathcal{P}$ and a lower bound on the maximum number of essentially different lines determined by $\mathcal{P}$.

Lemma 18. *There is a crossing family of size $\Omega(\mu(\mathcal{M}_{oc}))$ with vertices in $\mathcal{P}$.*

Proof. Place the point C (which is the center of the double-rectangles B_i) randomly according to the uniform distribution in the disk $D' = D\left(O, s+\frac{t}{2}\right)$ and choose the angle φ randomly according to the uniform distribution in the interval $[0, 2\pi)$. Then, for any $i = 1, \ldots, m$, the double-rectangle B_i is chosen randomly from the set $\mathcal{M}$ according to the distribution which corresponds to the measure μ. Thus, for every $i = 1, \ldots, m$, the double-rectangle B_i is occupied with probability $\mu(\mathcal{M}_{oc})/\mu(\mathcal{M})$. It follows that the expected number of occupied double-rectangles $B_i, i = 1, \ldots, m$, is $m \cdot \mu(\mathcal{M}_{oc})/\mu(\mathcal{M}) = \Theta(\mu(\mathcal{M}_{oc}))$. Consequently, there exist a point $C \in D'$ and an angle $\varphi \in [0, 2\pi)$ such that at least $\Omega(\mu(\mathcal{M}_{oc}))$ double-rectangles are occupied. An application of Observation 17 completes the proof. ∎

Lemma 19. *There are $\Omega(n \cdot \mu(\mathcal{M}_{oc}))$ pairwise essentially different lines determined by the set $\mathcal{P}$.*

Proof. Let $\mathcal{U}$ be an arbitrary maximal collection of pairwise essentially different lines determined by an α-dense set $\mathcal{P}$. For any $l \in \mathcal{U}$, let $\mathcal{M}_l$ be the set of all double-rectangles of $\mathcal{M}$ with direction different from the direction of l by at most $\frac{1}{n} + \arctan\left(\frac{t}{\sqrt{n}/4}\right) \le \frac{5}{n}$ and with center in the $\left(\frac{1}{\sqrt{n}} + \frac{5}{n} \cdot \alpha\sqrt{n}\right) = \left(\frac{1+5\alpha}{\sqrt{n}}\right)$-neighborhood of l. Then any pair of $\mathcal{P}^2$ which occupies any double-rectangle of $\mathcal{M} \setminus \mathcal{M}_l$ determines a line essentially different to l. Hence, by the maximality of $\mathcal{U}$,

$$\mathcal{M}_{oc} \subseteq \bigcup_{l\in\mathcal{U}} \mathcal{M}_l.$$

Clearly, for any $l \in \mathcal{U}$,

$$\mu(\mathcal{M}_l) \le 2\frac{1+5\alpha}{\sqrt{n}}\alpha\sqrt{n} \cdot 2\frac{5}{n} = \frac{20\alpha(5\alpha+1)}{n}.$$

Thus,

$$\mu(\mathcal{M}_{oc}) \le \mu\left(\bigcup_{l\in\mathcal{U}} \mathcal{M}_l\right) \le \sum_{l\in\mathcal{U}} \mu(\mathcal{M}_l) \le \frac{20\alpha(5\alpha+1)}{n}|\mathcal{U}|$$

and, consequently,

$$|\mathcal{U}| \ge \Omega(n \cdot \mu(\mathcal{M}_{oc})).$$ ∎

Our goal is to show that $\mu(\mathcal{M}_{oc}) \geq \Omega(n^{1-\varepsilon})$ which, applying Lemmas 19 and 18, immediately gives Theorems 15 and 16.

First we show two packing lemmas for sets with minimum separation 1 and give bounds on $\mu(\mathcal{M}(x,y))$ and on $\sum \mu(\mathcal{M}(x,y))$.

Lemma 20. *Let $\mathcal{P}$ be an arbitrary set of points in the plane with the minimum distance* 1. *Then, for every $r>0$, there are at most $\frac{2\pi}{\sqrt{3}}r^2+O(r)$ points of $\mathcal{P}$ in the r-neighborhood of any point p in the plane.*

Lemma 21. *Let $\mathcal{P}$ be an arbitrary set of points in the plane with the minimum distance* 1. *Then, for every positive integer j and for every point p in the plane, there are at most $44j$ points $a \in \mathcal{P}$ such that $\lfloor |pa| \rfloor = j$.*

Lemma 22. (i) *There is a positive constant c_1 such that, for any two points $x, y \in \mathcal{P}$,*

$$\mu(\mathcal{M}(x,y)) \leq \frac{c_1}{n}.$$

(ii)

$$\sum_{(x,y)\in\mathcal{P}^2\setminus T} \mu(\mathcal{M}(x,y)) = \Omega(n),$$

for any subset T of $\mathcal{P}^2$ of size at most $0.5n^2$.

Proof of Lemma 20. We shall use one of the basic results in the packing theory which says that, for any $R>0$, there are at most $\frac{\pi}{2\sqrt{3}}R^2+O(R)$ mutually non-overlaping disks of diameter 1 inside a disk of diameter R (e.g., see [7]).

The $\frac{1}{2}$-neighborhoods of points of $\mathcal{P}$ do not overlap. For each point $a \in \mathcal{P}$ lying in the r-neighborhood of p, the $\frac{1}{2}$-neighborhood of a lies entirely in the $(r+\frac{1}{2})$-neighborhood of p. Now the above packing result implies that the number of points of $\mathcal{P}$ lying in the r-neighborhood of p is at most $\frac{\pi}{2\sqrt{3}}(2r+1)^2+O(2r+1)=\frac{2\pi}{\sqrt{3}}r^2+O(r)$. ∎

Proof of Lemma 21. Let C_1, C_2, C_3 be circles centered at p with radii j, $(j+\frac{1}{2})$, $j+1$, respectively. All the points $a \in \mathcal{P}$ with $\lfloor |pa| \rfloor = j$ lie in the annulus A with border circles C_1 and C_3. The circles C_1, C_2, C_3 and the $11j$ lines of directions $\frac{i}{11j}\pi, i=0,1,\ldots,11j-1$, going through the point p partition A into $44j$ regions. The perimeter of each of these regions is $\frac{\pi}{11j}(2j+\frac{1}{2})+1$ or $\frac{\pi}{11j}\left(2j+\frac{3}{2}\right)+1$. Hence, it is smaller than 2 which implies that each of these $44j$ regions contains at most one point of $\mathcal{P}$. ∎

Proof of Lemma 22. (i) The centers of double-rectangles occupied by a fixed pair $(x,y)\in\mathcal{P}^2$ lie in the $\frac{t}{2}$-neighborhood of the line xy and in the s-neighborhood of the point x. Thus, they lie in a region whose area is bounded from above by a constant. Their directions may obviously vary by at most $2\arctan\left(\frac{t}{\sqrt{n}/4}\right) \le 2\frac{t}{\sqrt{n}/4} = \frac{8}{n}$. Statement (i) follows.

(ii) Lemma 20 implies that, for n large, there are at least $0.55n^2$ pairs $(x,y)\in\mathcal{P}^2$ with $|xy|\ge\frac{\sqrt{n}}{3}$. Hence, there are at least $0.05n^2$ pairs $(x,y)\in\mathcal{P}^2\setminus T$ with $|xy|\ge\frac{\sqrt{n}}{3}$.

Let $(x,y)\in\mathcal{P}^2\setminus T$ be any pair of $\mathcal{P}^2\setminus T$ with $|xy|\ge\frac{\sqrt{n}}{3}$, and let $S(x,y)$ be the set of all the points which lie in the $\frac{1}{2\sqrt{n}}$-neighborhood of the line segment xy and whose distance to each of the two points x and y is at least $\frac{\sqrt{n}}{8}+\frac{1}{\sqrt{n}}$. The area of $S(x,y)$ is bounded from below by a constant. Any double-rectangle with center in $S(x,y)$ and with direction which differs from the direction of the (oriented) line yx by at most $\frac{1}{2\alpha n}$ is occupied by the pair (x,y). Thus,

$$\mu(\mathcal{M}(x,y)) \ge \Omega\left(\frac{1}{n}\right),$$

which yields the statement. ∎

In the sum in Lemma 22 (ii), for $T=\emptyset$, each occupied double-rectangle is counted so many times how many pairs of $\mathcal{P}^2$ occupy it while in $\mu(\mathcal{M}_{oc})$ only once. For to prove $\mu(\mathcal{M}_{oc})\ge\Omega(n^{1-\varepsilon})$ we need to show, roughly saying, that not too many occupied double-rectangles are occupied by too many pairs of $\mathcal{P}^2$. This is actually the key part of the proofs of Theorems 15 and 16, and it consists of Lemmas 23–25 below.

For two points x, $y\in\mathcal{P}$ and for $\delta>0$, let $Q(x,y,\delta)$ be the rectangle of size $|xy|\times 2\delta$ whose sides of length 2δ have the mid-points x and y, respectively. For any $r,\delta>0$, set

$$T(r,\delta)=\{(x,y)\in\mathcal{P}^2 : x\ne y, Q(x,y,\delta)\cap(N(x,r)\setminus\{x\})\cap\mathcal{P}\ne\emptyset\},$$

where $N(x,r)$ is the closed r-neighborhood of x.

Here is our key lemma:

Lemma 23. *Let r, r', δ, δ' be four positive real numbers such that $r>r'>\delta'$ and $\delta'\le\frac{\alpha\sqrt{n}}{r}\delta$. Then*

$$|T(r,\delta)| < |T(r',\delta')| + 200\alpha^2\frac{n^2 r\delta}{\delta'(r'-\delta')}.$$

Specially, if $\delta'=\frac{\alpha\sqrt{n}}{r}\delta$, then

$$|T(r,\delta)| < |T(r',\delta')| + 200\alpha\frac{n^{3/2}r^2}{r'-\delta'}.$$

Proof. Fix a point $x \in \mathcal{P}$. For any pair $(x,y) \in T(r,\delta)$, there exists a point $z \in \mathcal{P}$ in $N(x,r)$ such that the point y lies in the intersection of the disk $D(x,\alpha\sqrt{n}) = D(x,s)$ with $W(x,z)$, where $W(x,z)$ is a wedge with vertex x, with axis xz, and with angle $2\arcsin(\min\{\delta/|xz|,1\}) \leq \pi\delta/|xz|$ (see Fig. 2). For any $z \in \mathcal{P}$ lying in $N(x,r)$, the sector $D(x,s) \cap W(x,z)$ can be covered by $\left\lceil \frac{s\cdot\pi\delta/|xz|}{\delta'} \right\rceil = \left\lceil \pi\frac{s\delta}{\delta'|xz|} \right\rceil \leq \frac{4s\delta}{\delta'|xz|}$ rectangles of size $s \times \delta'$. (The inequality $\left\lceil \pi\frac{s\delta}{\delta'|xz|} \right\rceil \leq \frac{4s\delta}{\delta'|xz|}$ follows from the inequality $\frac{s\delta}{\delta'|xz|} \geq 1$ which is a consequence of the assumption $\delta' \leq \frac{\alpha\sqrt{n}}{r}\delta$.) Moreover, the covering can be found so that one of the two sides of length δ' of any rectangle in the covering contains x (see Fig. 2). Lemma 21 yields now that the set $U(x) = \{y \in \mathcal{P} : (x,y) \in T(r,\delta)\}$ can be covered by at most $\sum_{j=1}^{\lfloor r \rfloor} 44j \cdot \frac{4s\delta}{\delta' j} < 200\frac{rs\delta}{\delta'}$ rectangles of size $s \times \delta'$. Denote these rectangles by $G(x,1)$, $G(x,2)$, ..., $G(x,g(x))$, where $g(x) < 200\frac{rs\delta}{\delta'}$.

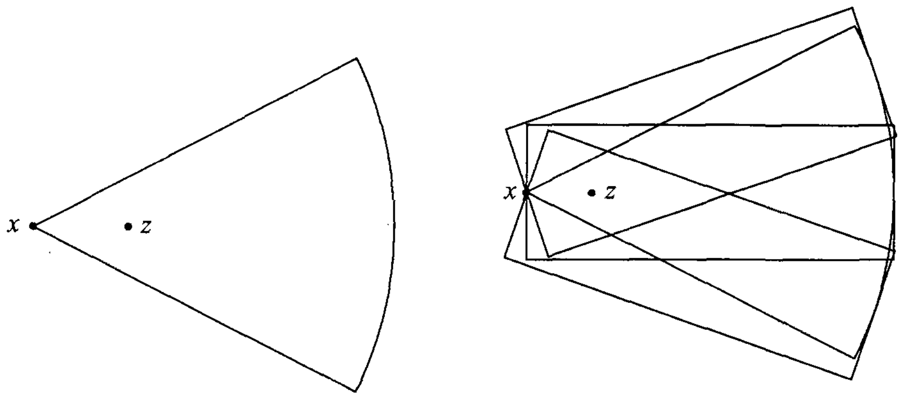

Fig. 2. Sector $D(x,s) \cap W(x,z)$ and its cover by (three) rectangles

Let $i \in \{1,2,\ldots,g(x)\}$. For any two different points $y_1, y_2 \in G(x,i)$, either $|y_1y_2| > r'$ or at least one of the pairs (y_1,x) and (y_2,x) lies in $T(r',\delta')$. Thus, the distance between any two points in the set $\{x\} \cup \{y \in G(x,i) : (y,x) \notin T(r',\delta')\}$ is bigger than r'. It follows that the size of the set $\{y \in G(x,i) : (y,x) \notin T(r',\delta')\}$ is smaller than $\frac{s}{\sqrt{(r')^2-(\delta')^2}} < \frac{s}{r'-\delta'}$. Thus, the set $U(x)$ contains fewer than $g(x)\frac{s}{r'-\delta'} < 200\frac{rs^2\delta}{\delta'(r'-\delta')}$ points y such that $(y,x) \notin T(r',\delta')$. In other words, there are at most $200\frac{rs^2\delta}{\delta'(r'-\delta')}$ points $y \in \mathcal{P}$ such that $(x,y) \in T(r,\delta)$ and $(y,x) \notin T(r',\delta')$. Since this holds for any $x \in \mathcal{P}$, we obtain

$$|T(r',\delta')| > |T(r,\delta)| - n \cdot 200\frac{rs^2\delta}{\delta'(r'-\delta')} = |T(r,\delta)| - 200\alpha^2\frac{n^2r\delta}{\delta'(r'-\delta')}.$$

The second part of Lemma 23 immediately follows from the first one. ∎

Lemma 24. *Let* $r > t = \frac{1}{\sqrt{n}}$ *and* $|T(r,t)| \le 0.25n^2$. *Then*

$$\mu(\mathcal{M}_{oc}) = \Omega\left(\frac{n}{\left(\frac{\alpha\sqrt{n}}{r-t}+1\right)^2}\right).$$

In particular, if $1.1t < r < \alpha\sqrt{n}$ *and* $|T(r,t)| \le 0.25n^2$, *then*

$$\mu(\mathcal{M}_{oc}) = \Omega\left(r^2\right).$$

Proof. Let $(x,y),(x',y') \in \mathcal{P}^2 \setminus T(r,t)$ be two pairs of points which occupy the same double-rectangle B and which do not lie in $T(r,t)$. Then either $x = x'$ or $|xx'| > r$. It follows that, for any double-rectangle B, there are fewer than $\frac{s}{\sqrt{r^2-t^2}} + 1 < \frac{s}{r-t} + 1$ points x such that, for at least one point $y \in \mathcal{P}$, the pair (x,y) occupies B and does not lie in $T(r,t)$. Similarly, there are at most $\frac{s}{r-t} + 1$ points y such that, for at least one point $x \in \mathcal{P}$, the pair (x,y) occupies B and (y,x) does not lie in $T(r,t)$. Thus, every double-rectangle is occupied by at most $\left(\frac{s}{r-t}+1\right)^2$ pairs of $\mathcal{P}^2 \setminus T$, where $T = \{(x,y) \in \mathcal{P}^2 : (x,y) \in T(r,t) \text{ or } (y,x) \in T(r,t)\}$. Therefore, the measure of occupied double-rectangles is at least

$$\mu(\mathcal{M}_{oc}) \ge \frac{\sum_{(x,y)\in\mathcal{P}^2\setminus T} \mu(\mathcal{M}(x,y))}{\left(\frac{s}{r-t}+1\right)^2} = \Omega\left(\frac{n}{\left(\frac{s}{r-t}+1\right)^2}\right).$$

The last inequality follows from Lemma 22 (ii) since $|T| \le 0.5n^2$.

The second part of Lemma 24 immediately follows from the first one. ∎

Lemma 25. *For any fixed* $\varepsilon > 0$,

$$\mu(\mathcal{M}_{oc}) \ge \Omega(n^{1-\varepsilon}).$$

Proof. Let $\alpha \ge \sqrt{2\sqrt{3}}/\pi$ be fixed. It suffices to prove the lemma for any $\varepsilon = \frac{1}{2^k} + \frac{1}{2^{4k-1}}$, $k = 1,2,3,\ldots$

Set $d_0 = e_0 = 1$ and, for $i = 1,\ldots,k$, define

$$d_i = 1600\alpha k(d_{i-1})^2, \qquad e_i = \frac{\alpha e_{i-1}}{d_{i-1}}.$$

Further, for $i = 0,1,\ldots,k$, define

$$r_i = d_i n^{\frac{1}{2}-2^{i-1}\varepsilon}, \qquad \delta_i = e_i n^{(2^{i-1}-\frac{1}{2})\varepsilon-\frac{1}{2}}.$$

For $i=0, \ldots, k-1$, we can apply Lemma 23 with $r=r_i$, $\delta=\delta_i$, $r'=r_{i+1}$, $\delta'=\delta_{i+1}$ since $r_i > r_{i+1} > \delta_{i+1}$ (provided n is sufficiently large) and $\delta_{i+1} = \frac{\alpha\sqrt{n}}{r_i}\delta_i$. We have the following estimate for the "difference" term in Lemma 23:

$$200\alpha\frac{n^{3/2}r^2}{r'-\delta'} \le 200\alpha\frac{n^{3/2}(d_i n^{\frac{1}{2}-2^{i-1}\varepsilon})^2}{d_{i+1}n^{\frac{1}{2}-2^i\varepsilon}/2} = 400\alpha\frac{d_i^2 n^2}{d_{i+1}} = \frac{n^2}{4k}.$$

Thus, for any sufficiently large n,

$$\begin{aligned}\left|T\left(n^{\frac{1}{2}-\frac{\varepsilon}{2}}, \frac{1}{\sqrt{n}}\right)\right| &< \left|T\left(d_1 n^{\frac{1}{2}-\varepsilon}, e_1 n^{\frac{\varepsilon}{2}-\frac{1}{2}}\right)\right| + \frac{n^2}{4k} < \\ &< \left|T\left(d_2 n^{\frac{1}{2}-2\varepsilon}, e_2 n^{\frac{3\varepsilon}{2}-\frac{1}{2}}\right)\right| + \frac{2n^2}{4k} < \\ &< |T(d_3 n^{\frac{1}{2}-4\varepsilon}, e_3 n^{(4-\frac{1}{2})\varepsilon-\frac{1}{2}})| + \frac{3n^2}{4k} < \\ &\quad \cdots\cdots\cdots \\ &< |T(d_k n^{\frac{1}{2}-2^{k-1}\varepsilon}, e_k n^{(2^{k-1}-\frac{1}{2})\varepsilon-\frac{1}{2}})| + \frac{kn^2}{4k} = \\ &= |T(d_k n^{-1/2^{3k}}, e_k n^{-1/2^{k+1}+1/2^{3k}-1/2^{4k}})| + \tfrac{n^2}{4} = \tfrac{n^2}{4}.\end{aligned}$$

Lemma 24 (with $r=n^{\frac{1}{2}-\frac{\varepsilon}{2}}$) yields now

$$\mu(\mathcal{M}_{oc}) \ge \Omega(n^{1-\varepsilon}). \qquad \blacksquare$$

Proof of Theorems 15 and 16.

Lemmas 19 and 25 yield Theorem 15. Lemmas 18 and 25 yield Theorem 16. ∎

3. A better bound

In this section we give the proofs of Theorems 8 and 9. According to Lemmas 18 and 19 Theorems 8 and 9 immediately follow from the following lemma.

Lemma 26. *For any fixed* $\varepsilon > 0$,

$$\mu(\mathcal{M}_{oc}) \ge \Omega\left(\frac{n}{(\log n)^{4\log\log\log n + c}}\right),$$

where $c = 4\log(1000\alpha^2)+8$.

Now, in the proof of Lemma 26 we do a more precise calculation than in the previous section.

Proof of Lemma 26. We repeatedly apply Lemma 23 as in the proof of Lemma 25. However, now we apply it more than a constant number of times. Throughout the whole proof n is a sufficiently large positive integer.

Set $k = \lfloor \log\log n - \log\log\log n - \log(\log\log\log n + z + 1) - 1 \rfloor$, where $z = \log(1000\alpha^2)$. The number k shall be the number of applications of Lemma 23. It is interesting that it is important to take the number k with so large precision. Otherwise we would not obtain the desired bound. The definition of k yields the following estimate which shall be useful later:

$$\frac{2\log\log n(\log\log\log n + z + 1)}{\log n} \le \frac{1}{2^k} < \frac{4\log\log n(\log\log\log n + z + 1)}{\log n}.$$

Set $r_0 = \frac{9}{10}$ and define numbers $r_1, r_2, \dots, r_k$ inductively by

$$r_i = \sqrt{\frac{\sqrt{n}}{1000\alpha k} r_{i-1}}, \quad \text{for } i = 1, 2, \dots, k.$$

Further set $\delta_0 = \frac{1}{10}$ and define numbers $\delta_1, \delta_2, \dots, \delta_k$ inductively by

$$\delta_i = \frac{r_i}{\alpha\sqrt{n}} \delta_{i-1}, \quad \text{for } i = 1, 2, \dots, k.$$

Later we shall apply Lemma 23 for $i = 1, 2, \dots, k$ with $r = r_i$, $\delta = \delta_i$, $r' = r_{i-1}$, and $\delta' = \delta_{i-1}$. Now we express numbers $r_i\,(i = 1, 2, \dots, k)$ and $\delta_i\,(i = 1, 2, \dots, k)$ in an explicit form, and then estimate the numbers r_k and δ_k and show what we get by repeated applications of Lemma 23.

For any $i = 1, 2, \dots, k$,

$$r_i = \left(\frac{\sqrt{n}}{1000\alpha k}\right)^{\frac{1}{2}} r_{i-1}^{\frac{1}{2}} = \left(\frac{\sqrt{n}}{1000\alpha k}\right)^{\frac{1}{2}} \left(\frac{\sqrt{n}}{1000\alpha k}\right)^{\frac{1}{4}} r_{i-2}^{\frac{1}{4}} = \dots =$$

$$= \left(\frac{\sqrt{n}}{1000\alpha k}\right)^{1-\frac{1}{2^i}} r_0^{\frac{1}{2^i}}$$

and

$$\delta_i = \frac{r_i r_{i-1} \cdots r_1}{(\alpha\sqrt{n})^i} \delta_0 =$$

$$= \frac{\left(\frac{\sqrt{n}}{1000\alpha k}\right)^{1-\frac{1}{2^i}} r_0^{\frac{1}{2^i}} \cdot \left(\frac{\sqrt{n}}{1000\alpha k}\right)^{1-\frac{1}{2^{i-1}}} r_0^{\frac{1}{2^{i-1}}} \cdots \left(\frac{\sqrt{n}}{1000\alpha k}\right)^{1-\frac{1}{2}} r_0^{\frac{1}{2}}}{(\alpha\sqrt{n})^i} \delta_0 =$$

$$= \frac{\left(\frac{\sqrt{n}}{1000\alpha k}\right)^{i-1+\frac{1}{2^i}}}{(\alpha\sqrt{n})^i} r_0^{1-\frac{1}{2^i}} \delta_0.$$

Thus,

$$r_k = \left(\frac{\sqrt{n}}{1000\alpha k}\right)^{1-\frac{1}{2^k}} r_0^{\frac{1}{2^k}} > \frac{\sqrt{n}}{1000\alpha k}(\sqrt{n})^{-\frac{1}{2^k}} >$$

$$> \frac{\sqrt{n}}{1000\alpha 2^{\log\log\log n}} \left(2^{\frac{\log n}{2}}\right)^{-\frac{4\log\log n\,(\log\log\log n+z+1)}{\log n}} \geq$$

$$\geq \frac{\sqrt{n}}{2^{2\log\log n(\log\log\log n+z+2)}} = \frac{\sqrt{n}}{(\log n)^{2\log\log\log n+2z+4}}$$

and

$$\delta_k = \frac{\left(\frac{\sqrt{n}}{1000\alpha k}\right)^{k-1+\frac{1}{2^k}}}{(\alpha\sqrt{n})^k} r_0^{1-\frac{1}{2^k}}\delta_0 = \frac{(\sqrt{n})^{\frac{1}{2^k}}}{(1000\alpha^2 k)^{k-1+\frac{1}{2^k}}} \cdot \frac{1}{\sqrt{n}}\alpha^{\frac{1}{2^k}-1} r_0^{1-\frac{1}{2^k}}\delta_0 =$$

$$= \frac{\exp\frac{\log n}{2}\cdot\frac{1}{2^k}}{\exp\left(k-1+\frac{1}{2^k}\right)\log(1000\alpha^2 k)} \cdot \frac{1}{\sqrt{n}}\alpha^{\frac{1}{2^k}-1} r_0^{1-\frac{1}{2^k}}\delta_0 \geq$$

$$\geq \frac{\exp\frac{\log n}{2}\cdot\frac{2\log\log n(\log\log\log n+z+1)}{\log n}}{\exp\log\log n\cdot\log(1000\alpha^2\log\log n)} \cdot \frac{1}{\sqrt{n}}\alpha^{\frac{1}{2^k}-1} r_0^{1-\frac{1}{2^k}}\delta_0 =$$

$$= \frac{\exp\log\log n(\log\log\log n+z+1)}{\exp\log\log n(\log(1000\alpha^2)+\log\log\log n)} \cdot \frac{1}{\sqrt{n}}\alpha^{\frac{1}{2^k}-1} r_0^{1-\frac{1}{2^k}}\delta_0 =$$

$$= \exp\log\log n \cdot \frac{1}{\sqrt{n}}\alpha^{\frac{1}{2^k}-1} r_0^{1-\frac{1}{2^k}}\delta_0 \geq \frac{1}{\sqrt{n}}.$$

Note now that the number k was determined as large as possible so that the above estimate $\delta_k \geq \frac{1}{\sqrt{n}}$ still holds.

The inductive definition of numbers δ_i $(i=1, \ldots, k)$ and the explicit expression of numbers r_i $(i=1, \ldots, k)$ immediately give, for any $i=1,\ldots,k$, the assumptions of Lemma 23 with $r=r_i$, $\delta=\delta_i$, $r'=r_{i-1}$, and $\delta'=\delta_{i-1}$. Repeated applications of Lemma 23 give

$$|T(r_k,\delta_k)| < |T(r_{k-1},\delta_{k-1})| + 200\alpha\frac{n^{3/2}r_k^2}{r_{k-1}-\delta_{k-1}} =$$

$$= |T(r_{k-1},\delta_{k-1})| + 200\alpha\frac{n^{3/2}\frac{\sqrt{n}}{1000\alpha k}r_{k-1}}{r_{k-1}-\delta_{k-1}} <$$

$$< |T(r_{k-1},\delta_{k-1})| + \frac{0.25n^2}{k} <$$

$$< |T(r_{k-2},\delta_{k-2})| + 200\alpha\frac{n^{3/2}\frac{\sqrt{n}}{1000\alpha k}r_{k-2}}{r_{k-2}-\delta_{k-2}} + \frac{0.25n^2}{k} <$$

$$< |T(r_{k-2}, \delta_{k-2})| + 2 \cdot \frac{0.25n^2}{k} <$$

$$\cdots\cdots$$

$$< |T(r_0, \delta_0)| + k \cdot \frac{0.25n^2}{k} = 0.25n^2.$$

It follows that

$$\left| T\left(\frac{\sqrt{n}}{(\log n)^{2\log\log\log n + 2z + 4}}, \frac{1}{\sqrt{n}} \right) \right| \le |T(r_k, \delta_k)| < 0.25n^2.$$

Lemma 24 (with $r = \frac{\sqrt{n}}{(\log n)^{2\log\log\log n + 2z + 4}}$) yields now

$$\mu(\mathcal{M}_{oc}) \ge \Omega\left(\frac{n}{(\log n)^{4\log\log\log n + 4z + 8}} \right). \qquad \blacksquare$$

Proof of Theorems 8 and 9. Theorem 8 follows from Lemmas 19 and 26. Lemmas 18 and 26 yield Theorem 9. ∎

All the calculation giving the proofs of Theorems 8 and 9 was very precise. We do not expect that any approach based on the idea of Lemma 23 can give a substantially better bound in some of these two theorems. A tiny improvement upon the bound in Theorem 9 was shown in the author's PhD. thesis [12].

Theorem 27. ([12]) *For every fixed $\alpha > 0$, any α-dense set of size n contains vertices of a crossing family of size $\Omega\left(\frac{n}{(\log n)^{3\log\log\log n + c}}\right)$, where $c = 3\log(1000\alpha^2) + 6$.*

Theorem 27 relies on the following lemma which improves the procedure of Lemmas 18 and 24.

Lemma 28. ([12]) *If $1.1t < r < \alpha\sqrt{n}$ and $|T(r,t)| \le 0.25n^2$, then there is a crossing family of size $\Omega(r^{3/2}n^{1/4})$ with vertices in $\mathcal{P}$.*

Note that Lemmas 18 and 24 imply a weaker version of Lemma 28 when $\Omega(r^{3/2}n^{1/4})$ is replaces by $\Omega(r^2)$. We actually apply Lemmas 18 and 24 in the above proofs of Theorems 16 and 26 using exactly this fact. The refinement giving Lemma 28 is done by adding more than one line segment connecting points of the two rectangles of a double-rectangle B_i to the crossing family whenever B_i is occupied by a sufficiently large number of pairs of points. The improvement is very small in our case but we believe that the refinement might be more crucial with another method than our one.

We do not prove Theorem 27 and Lemma 28 here. The reader can find the proofs in [12].

4. Line-point incidences

In this section we prove Theorem 7 and Corollary 10. For $i=1,2,\ldots$, we define

$$T_i(r,\delta)=\{(x,y)\in T(r,\delta):\frac{\alpha\sqrt{n}}{2^i}<|xy|\le\frac{\alpha\sqrt{n}}{2^{i-1}}\}.$$

Note that $T(r,\delta)$ is a disjoint union of sets $T_i(r,\delta)$, $i=1,\,2,\,\ldots,\,\lfloor 2+\log(\alpha\sqrt{n})\rfloor=\lfloor\log(4\alpha\sqrt{n})\rfloor$.

The proof of Theorem 7 relies on the following extension of Lemma 23.

Lemma 29. *Let i be a positive integer, and let r, r', δ, δ' be four positive real numbers such that $r>r'>\delta'$ and $\delta'\le\frac{\alpha\sqrt{n}}{2^{i-1}r}\delta$. Then*

$$|T_i(r,\delta)|<|T_i(r',\delta')|+200\alpha^2\frac{n^2r\delta}{2^{i-1}\delta'(r'-\delta')}.$$

Specially, if $\delta'=\frac{\alpha\sqrt{n}}{r}\delta$, then

$$|T_i(r,\delta)|<|T_i(r',\delta')|+200\alpha\frac{n^{3/2}r^2}{2^{i-1}(r'-\delta')}.$$

Proof. We modify the proof of Lemma 23. Fix a point $x\in\mathcal{P}$. For any pair $(x,y)\in T_i(r,\delta)$, there exists a point $z\in\mathcal{P}$ in $N(x,r)$ such that the point y lies in the intersection of the disk $D\left(x,\frac{\alpha\sqrt{n}}{2^{i-1}}\right)=D(x,s^*)$ with $W(x,z)$. For any $z\in\mathcal{P}$ lying in $N(x,r)$, the sector $D(x,s^*)\cap W(x,z)$ can be covered by $4\frac{s\delta}{\delta'|xz|}$ rectangles of size $s^*\times\delta'$. The set $U(x)=\{y\in\mathcal{P}:(x,y)\in T_i(r,\delta)\}$ can be covered by fewer than $\sum_{i=1}^{\lfloor r\rfloor}44i\cdot\left(4\frac{s\delta}{\delta'|xz|}\right)<200\frac{rs\delta}{\delta'}$ rectangles of size $s^*\times\delta'$. Denote these rectangles by $G(x,1),G(x,2),\ldots,G(x,g(x))$, where $g(x)<200\frac{rs\delta}{\delta'}$.

Let $i\in\{1,2,\ldots,g(x)\}$. The size of the set $\{y\in G(x,i):(y,x)\notin T_i(r',\delta')\}$ is at most $\left\lceil\frac{s^*}{\sqrt{(r')^2-(\delta')^2}}\right\rceil-1<\frac{s^*}{r'-\delta'}$. Thus, the set $U(x)$ contains fewer than $g(x)\frac{s^*}{r'-\delta'}<200\frac{rs\delta s^*}{\delta'(r'-\delta')}$ points y such that $(y,x)\notin T_i(r',\delta')$. Since this holds for any $x\in\mathcal{P}$, we obtain the desired inequality

$$|T_i(r',\delta')|>|T_i(r,\delta)|-n\cdot 200\frac{rs\delta s^*}{\delta'(r'-\delta')}=|T_i(r,\delta)|-200\alpha^2\frac{n^2r\delta}{2^{i-1}\delta'(r'-\delta')}.$$

The second part of Lemma 29 immediately follows from the first one. ∎

Proof of Theorem 7. Let the assumptions of Theorem 7 be satisfied. Let p be the number of rough incidences between points in $\mathcal{P}$ and lines in $\mathcal{L}$. The estimate $p = O(m\sqrt{n})$ follows from the fact that any line is roughly incident to at most $O(\sqrt{n})$ points. The estimate $p = O\left(n^2\right)$ follows from the fact that any point is roughly incident to at most $O(n)$ essentially different lines. It remains to show that $p = O(\min\{m^{3/4}n^{5/8}\log^{1/4} n, n\sqrt{m}\})$. We may and shall assume that $p \geq 5m$ and $n \geq 100$ since otherwise the theorem certainly holds.

For any line $l \in \mathcal{L}$, all the points which are roughly incident to l lie in a rectangle R_l of size $\alpha\sqrt{n} \times \frac{2}{\sqrt{n}}$. For any line $l \in \mathcal{L}$, let N_l be the number of points of $\mathcal{P}$ lying in R_l. Certainly,

$$\sum_{l \in \mathcal{L}} N_l \geq p.$$

Any rectangle $R_l, l \in \mathcal{L}$, can be partitioned into $\lceil \frac{p}{2m} \rceil$ rectangles of size $\frac{\alpha\sqrt{n}}{\lceil \frac{p}{2m} \rceil} \times \frac{2}{\sqrt{n}}$. For any $y \in R_l \cap \mathcal{P}$, there exists at most one point $x \in R_l \cap \mathcal{P}$ in each of these $\lceil \frac{p}{2m} \rceil$ rectangles such that $(x,y) \notin T\left(\sqrt{\left(\frac{\alpha\sqrt{n}}{\lceil \frac{p}{2m} \rceil}\right)^2 + \left(\frac{2}{\sqrt{n}}\right)^2}, \frac{2}{\sqrt{n}}\right)$. Thus,

$$\left|(R_l \times R_l) \cap T\left(\sqrt{\left(\frac{\alpha\sqrt{n}}{\lceil \frac{p}{2m} \rceil}\right)^2 + \left(\frac{2}{\sqrt{n}}\right)^2}, \frac{2}{\sqrt{n}}\right)\right| \geq N_l\left(N_l - \left\lceil \frac{p}{2m} \right\rceil\right).$$

Since $\sqrt{\left(\frac{\alpha\sqrt{n}}{\lceil \frac{p}{2m} \rceil}\right)^2 + \left(\frac{2}{\sqrt{n}}\right)^2} \leq \frac{3\alpha m\sqrt{n}}{p}$, we get

$$\left|(R_l \times R_l) \cap T\left(\frac{3\alpha m\sqrt{n}}{p}, \frac{2}{\sqrt{n}}\right)\right| \geq N_l\left(N_l - \left\lceil \frac{p}{2m} \right\rceil\right).$$

Hence,

$$\sum_{l \in \mathcal{L}} \left|(R_l \times R_l) \cap T\left(\frac{3\alpha m\sqrt{n}}{p}, \frac{2}{\sqrt{n}}\right)\right| \geq \sum_{l \in \mathcal{L}} N_l\left(N_l - \left\lceil \frac{p}{2m} \right\rceil\right) \geq$$
$$\geq m\frac{p}{m}\left(\frac{p}{m} - \left\lceil \frac{p}{2m} \right\rceil\right) > \frac{p^2}{3m}.$$

Now we prove that $p = O(n\sqrt{m})$, then we will show that

$$p = O(m^{3/4}n^{5/8}\log^{1/4} n).$$

Since $T\left(\frac{3\alpha m\sqrt{n}}{p}, \frac{2}{\sqrt{n}}\right)$ is a disjoint union of sets $T_i\left(\frac{3\alpha m\sqrt{n}}{p}, \frac{2}{\sqrt{n}}\right)$, $i=1, 2, 3, \ldots, \lfloor\log(4\alpha\sqrt{n})\rfloor$, and since $\sum_{i=1}^{\infty} \frac{1}{2i^2} < 1$, there exists i, $0 < i \leq \lfloor\log(4\alpha\sqrt{n})\rfloor$, such that

$$\sum_{l\in\mathcal{L}} \left|(R_l \times R_l) \cap T_i\left(\frac{3\alpha m\sqrt{n}}{p}, \frac{2}{\sqrt{n}}\right)\right| > \frac{1}{2i^2} \cdot \frac{p^2}{3m} = \frac{p^2}{6i^2 m}.$$

Since every pair of points of $\mathcal{P}$ with distance bigger than $\frac{\alpha\sqrt{n}}{2^i}$ is contained in at most $\left\lceil n \cdot 2\arcsin\frac{2/\sqrt{n}}{\alpha\sqrt{n}/2^i}\right\rceil \leq \left\lceil\frac{2^{i+2}\pi}{\alpha}\right\rceil < 2^{i+4}$ common rectangles $R_l, l \in \mathcal{L}$, we have

$$\left|T_i\left(\frac{3\alpha m\sqrt{n}}{p}, \frac{2}{\sqrt{n}}\right)\right| > \frac{1}{2^{i+4}} \cdot \frac{p^2}{6i^2 m}.$$

According to Lemma 20,

$$\left|T_i\left(\frac{3\alpha m\sqrt{n}}{p}, \frac{2}{\sqrt{n}}\right)\right| \leq \left|\left\{(x,y) \in \mathcal{P} : |xy| \leq \frac{\alpha\sqrt{n}}{2^{i-1}}\right\}\right| \leq$$
$$\leq \left(\frac{2\pi}{\sqrt{3}} \cdot \frac{\alpha^2 n}{2^{2(i-1)}} + \frac{O(\sqrt{n})}{2^i}\right) = \frac{O(n^2)}{2^{2i}}.$$

Hence,

$$\frac{1}{2^{i+4}} \cdot \frac{p^2}{6i^2 m} \leq \frac{O(n^2)}{2^{2i}},$$
$$p^2 \leq \frac{i^2}{2^i} O(n^2 m),$$
$$p = O(n\sqrt{m}).$$

Now we prove that $p = O(m^{3/4} n^{5/8} \log^{1/4} n)$. We again use the above inequality

$$\sum_{l\in\mathcal{L}} \left|(R_l \times R_l) \cap T\left(\frac{3\alpha m\sqrt{n}}{p}, \frac{2}{\sqrt{n}}\right)\right| > \frac{p^2}{3m}.$$

Since $T\left(\frac{3\alpha m\sqrt{n}}{p}, \frac{2}{\sqrt{n}}\right)$ is a disjoint union of sets $T_i\left(\frac{3\alpha m\sqrt{n}}{p}, \frac{2}{\sqrt{n}}\right)$, $i=1, 2, 3, \ldots, \lfloor\log(4\alpha\sqrt{n})\rfloor$, there exists i, $0 < i \leq \lfloor\log(4\alpha\sqrt{n})\rfloor$, such that

$$\sum_{l\in\mathcal{L}} \left|(R_l \times R_l) \cap T_i\left(\frac{3\alpha m\sqrt{n}}{p}, \frac{2}{\sqrt{n}}\right)\right| > \frac{p^2}{3m(\log(4\alpha\sqrt{n}))}.$$

Since every pair of points of $\mathscr{P}$ with distance bigger than $\frac{\alpha\sqrt{n}}{2^i}$ is contained in at most 2^{i+4} common rectangles $R_l, l \in \mathscr{L}$, we have, for any $p^*, 0 < p^* \le p$,

$$\left|T_i\left(\frac{3\alpha m\sqrt{n}}{p^*}, \frac{2}{\sqrt{n}}\right)\right| \ge \left|T_i\left(\frac{3\alpha m\sqrt{n}}{p}, \frac{2}{\sqrt{n}}\right)\right| > \frac{1}{2^{i+4}} \cdot \frac{p^2}{3m(\log(4\alpha\sqrt{n}))}.$$

Define $p^* = \min\{p, 0.3m\sqrt{n}\}$. Lemma 29 (with $r = \frac{3\alpha m\sqrt{n}}{p^*}$, $r' = 0.8$, $\delta = \frac{2}{\sqrt{n}}$, $\delta' = \frac{\alpha\sqrt{n}}{r}\delta = \frac{2p^*}{3m\sqrt{n}}$) gives

$$\left|T_i\left(\frac{3\alpha m\sqrt{n}}{p^*}, \frac{2}{\sqrt{n}}\right)\right| \le |T_i(r',\delta')| + \frac{O\left(n^{3/2}r^2\right)}{2^i}.$$

Since $r' = 0.8$ and $\delta' \le 0.2$, we have $\sqrt{(r')^2 + (\delta')^2} < 1$. Therefore, the set $T(r',\delta')$ is empty. Hence,

$$\frac{1}{2^{i+4}} \cdot \frac{p^2}{3m(\log(4\alpha\sqrt{n}))} < \left|T_i\left(\frac{3\alpha m\sqrt{n}}{p^*}, \frac{2}{\sqrt{n}}\right)\right| \le \frac{O\left(n^{3/2}r^2\right)}{2^i} = \frac{O\left(\frac{m^2 n^{5/2}}{(p^*)^2}\right)}{2^i},$$

$$p^2(p^*)^2 \le O(m^3 n^{5/2}\log n).$$

If $p^* = p$ then

$$p = O\left((m^3 n^{5/2}\log n)^{1/4}\right) = O(m^{3/4}n^{5/8}\log^{1/4} n).$$

Otherwise $p^* = 0.3m\sqrt{n}$ and

$$p^2 = O\left(\frac{m^3 n^{5/2}\log n}{m^2 n}\right),$$

$$p = O(m^{1/2}n^{3/4}\sqrt{\log n}),$$

$$p = O(\min\{m^{1/2}n^{3/4}\sqrt{\log n}, m\sqrt{n}\}) \le O\left(\left(m^{1/2}n^{3/4}\sqrt{\log n}\right)^{1/2}\left(m\sqrt{n}\right)^{1/2}\right) =$$

$$= O\left(m^{3/4}n^{5/8}\log^{1/4} n\right). \qquad \blacksquare$$

Let us note that, for $n^{7/6}\log n < m < n^2$, repeated applications of Lemma 29 give bounds on the number of rough incidences which are better than the bound in Theorem 7. The proof of these bounds is technically complicated. Therefore we state the bounds without the proofs. For $n^{7/6}\log n < m < n^{3/2}$, two applications of Lemma 29 give the bound $O(m^{3/5}n^{4/5}\log^{2/5} n)$. For $n^{3/2} \le m < n^2$, repeated applications of Lemma 29 give the bound $O\left(n^{1-\frac{\varepsilon}{4\log\frac{1}{\varepsilon}+7}}\sqrt{m}\right)$, where $\varepsilon > 0$ is such that $m = n^{2-\varepsilon}$.

Proof of Corollary 10. According to Theorem 7, the number of rough incidences between points in $\mathcal{P}$ and lines in $\mathcal{L}$ is at most

$$O(\min\{m^{3/4}n^{5/8}\log^{1/4} n, n\sqrt{m}\}) \leq$$

$$\leq O(\left(m^{3/4}n^{5/8}\log^{1/4} n\right)^{4/5}\left(n\sqrt{m}\right)^{1/5}) =$$

$$= O(m^{7/10}n^{7/10}\log^{1/5} n).$$

This bound and the bound in Theorem 7 immediately give the desired bound. ∎

5. Related results

In this section we prove Theorems 12–14.

Proof of Theorem 12. Denote $n' = n/2$. The diameter of the α-dense set $\mathcal{P}$ is at most $\alpha\sqrt{n} = \sqrt{2}\alpha\sqrt{n'}$. Let $\mathcal{B} \subset \mathcal{P}, |\mathcal{B}| = n'$, be the set of blue points, and let $\mathcal{R} \subset \mathcal{P}, |\mathcal{R}| = n'$, be the set of red points. We modify the proof of Theorem 8 so that we always take the set $\mathcal{B}\times\mathcal{R}$ of size n' instead of $\mathcal{P}^2$ (thus considering pairs of $\mathcal{B}\times\mathcal{R}$ instead of pairs of $\mathcal{P}^2$) and replace n by n' and α by $\sqrt{2}\alpha$. Correspondingly, in a proper way we can modify the whole proof of Theorem 8 so that it gives the desired bound $\Omega\left(\frac{(n')^2}{(\log(n'))^{4\log\log\log(n')+c}}\right) = \Omega\left(\frac{n^2}{(\log n)^{4\log\log\log n+c}}\right)$ in Theorem 12. ∎

Proof of Theorem 13. First we show a deterministic algorithm for a weaker (uncolored) version of Theorem 13. We shall turn the above proof of Theorem 9 into an algorithm. Define rectangles $R_i, i = 1, 2, \ldots, 2m$, as in Section 2. For $i = 1, 2, \ldots, 2m$, let $\bar{R}_i$ be the rectangle of size $\left(\alpha\sqrt{n} - \frac{1}{5\sqrt{n}}\right) \times \frac{4}{5\sqrt{n}}$ inscribed to R_i whose center coincides with the center of R_i and whose sides are parallel to the sides of R_i. Thus, $\bar{R}_i$ is the rectangle R_i without a boundary "frame" of width $\frac{1}{10\sqrt{n}}$. Analogously as in Section 2, we can define when a pair $\bar{B}_i = (\bar{R}_i, \bar{R}_{m+i}), i \in \{1, 2, \ldots, m\}$, is occupied. It is not difficult to see that using the methods from the previous section one can prove that there is a point $C = C_0$ and an angle $\varphi = \varphi_0$ such that the number of occupied pairs $\bar{B}_i$ is at least

$$\Omega\left(\frac{n}{(\log n)^{4\log\log\log n+c}}\right).$$

Then, whenever C lies in the $\frac{1}{100\sqrt{n}}$-neighborhood of C_0 and φ differs from φ_0 by at most $\frac{1}{100\alpha n}$, the number of occupied double-rectangles is at least

$$\Omega\left(\frac{n}{(\log n)^{4\log\log\log n+c}}\right).$$

It follows that there exist a point $C \in \{(\frac{i}{100\sqrt{n}}, \frac{j}{100\sqrt{n}}) : i,j \text{ integers}\}$ (which is the center of the double-rectangles B_i) and an angle $\varphi \in \{\frac{1}{g}2\pi, \frac{2}{g}2\pi, \ldots, 2\pi\}$, where $g = 2m\left\lceil\frac{100\alpha n}{2m}2\pi\right\rceil$, such that the number of occupied double-rectangles is at least $\Omega\left(\frac{n}{(\log n)^{4\log\log\log n+c}}\right)$. Certainly, the number of occupied double-rectangles does not change if we replace φ by $\varphi + \frac{i}{2m}2\pi$, where i is any integer. It follows that there exist a point $C \in \left\{\left(\frac{j}{100\sqrt{n}}, \frac{k}{100\sqrt{n}}\right) : j,k \text{ integers}\right\}$ and an angle $\varphi \in \{\frac{1}{g}2\pi, \frac{2}{g}2\pi, \ldots, \frac{1}{2m}2\pi\}$ such that at least $\Omega\left(\frac{n}{(\log n)^{4\log\log\log n+c}}\right)$ double-rectangles B_i are occupied.

Set $S_C = D' \cap \left\{\left(\frac{j}{100\sqrt{n}}, \frac{k}{100\sqrt{n}}\right) : j,k \text{ integers}\right\}$ and $S_\varphi = \{\frac{1}{g}2\pi, \frac{2}{g}2\pi, \ldots, \frac{1}{2m}2\pi\}$. We shall show that it is possible to find a point $C \in S_C$ and an angle $\varphi \in S_\varphi$ maximizing the number of occupied double-rectangles in time $O(n^3)$.

For simplicity, extend S_C so that

$$S_C = \left\{\left(\frac{j}{100\sqrt{n}}, \frac{k}{100\sqrt{n}}\right) : j \in \{j_0, j_0+1, \ldots, j_1\}, k \in \{k_0, k_0+1, \ldots, k_1\}\right\},$$

where j_0, j_1, k_0, k_1 are four integers such that $j_1 - j_0 = k_1 - k_0 = \lceil \alpha\sqrt{n} \cdot 100\sqrt{n} \rceil$. Note that $|S_C| = O(n^2)$ and $|S_\varphi| = O(1)$. For every $C \in S_C$, we can split the plane into $2m+1$ regions $Q_0, Q_1, \ldots, Q_{2m}$ as follows:

$$Q_0 = \left\{x \in \mathbb{R}^2 : |xC| < \frac{\sqrt{n}}{8}\right\},$$

$$Q_i = \left\{x \in \mathbb{R}^2 : |xC| \geq \frac{\sqrt{n}}{8}, \text{ the direction of the vector } \vec{Cx} \text{ lies in the interval } \left[\varphi + \frac{i-\frac{1}{2}}{2m}2\pi, \varphi + \frac{i+\frac{1}{2}}{2m}2\pi\right)\right\},$$

for $i = 1, \ldots, 2m$. Thus, for $i = 1, \ldots, 2m$, the rectangle R_i lies in the region Q_i.

For every $\varphi \in S_\varphi$ and for every $j = j_0, j_0+1, \ldots, j_1$, we perform the procedure described in the following three paragraphs.

We shall use $4m+2$ variables $F_0, F_0', F_1, F_1', \ldots, F_{2m}, F_{2m}'$ for subsets of $\mathcal{P}$, variable *maxvalue* for an integer, and variable *maxcenter* for a point in the plane.

At the beginning, set $F_0 = \mathcal{P}$, $F_1 = F_2 = \ldots = F_{2m} = \emptyset$, and $maxvalue = 0$. For $k = k_0, k_0+1, \ldots, k_1$, we perform the procedure described in the following two paragraphs. (Thus, the following two paragraphs will be performed $O(n^2)$ times.)

Set $C = \left(\frac{j}{100\sqrt{n}}, \frac{k}{100\sqrt{n}}\right)$ and define regions Q_0, Q_1, ..., Q_{2m} as described above. For every $a \in F_0$, establish whether a lies in Q_0. If $a \in Q_0$, put a to F'_0. Otherwise find the region Q_i, $i=1,\ldots,2m$, such that $a \in Q_i$, and put a to F'_i. For every $i = 1,2,\ldots,2m$ and every $a \in F_i$, we establish in which of the four regions $Q_0, Q_{i-1}, Q_i, Q_{i+1}$ the point a lies, and put a to the corresponding of the four sets F'_0, F'_{i-1}, F'_i, F'_{i+1}.

Afterwards, for every $i = 0,1,\ldots,2m$, set $F_i := F'_i$ and $F'_i := \emptyset$. If the number of occupied double-rectangles is bigger than *maxvalue* then put this number to *maxvalue* and put C to *maxcenter*. (This paragraph can be done in time $O(n)$.)

According to the proof of Theorem 9, the above procedure finds a point $C \in S_C$ and an angle $\varphi \in S_\varphi$ such that at least $\Omega\left(\frac{n}{(\log n)^{4\log\log\log n+c}}\right)$ double-rectangles B_i are occupied. Following Theorem 3 and the proof of Theorem 9 one can construct a crossing family of size at least $\Omega\left(\frac{n}{(\log n)^{4\log\log\log n+c}}\right)$.

The proof that the algorithm runs in time $O(n^3)$ is based on the above two notes in the brackets and on the fact that, for a fixed $j \in \{j_0,\ldots,j_1\}$, any point $a \in \mathcal{P}$ lies in F_0 and not in Q_0 for at most two different values of k (one of them is $k = k_0$). The details are left to the reader.

Using similar arguments as in the proof of Theorem 12, we can modify the above algorithm so that it works for the colored case as required.

Now we explain the randomized algorithm. The idea is very easy. We choose randomly a point $C \in D'$ and an angle $\varphi \in [0, 2\pi)$ and find a crossing family following the procedure above. With a probability bigger than p, where $p > 0$ is a positive constant, we obtain a crossing family of size at least $\Omega\left(\frac{n}{(\log n)^{4\log\log\log n+c}}\right)$. It is not difficult to see that this can be done in time $O(n\log n)$. If we obtain a crossing family of size smaller than $\Omega\left(\frac{n}{(\log n)^{4\log\log\log n+c}}\right)$, we again choose randomly a point $C \in D'$ and an angle $\varphi \in [0, 2\pi)$ and find a crossing family following the procedure above. We repeat this until we obtain a crossing family of size at least $\Omega\left(\frac{n}{(\log n)^{4\log\log\log n+c}}\right)$.

Obviously, the algorithm terminates in expected time $O(n\log n)$. ∎

Note that in the randomized algorithm described in the above proof we need to know a concrete function instead of the bound $\Omega\left(\frac{n}{(\log n)^{4\log\log\log n+c}}\right)$ in Theorem 9. This is not the case in the deterministic algorithm. Actually, we conjecture but have no proof that the deterministic algorithm in the above proof finds a crossing family of linear size.

Proof of Theorem 14. (i) Let $\mathcal{P}$ be a set of n points which are randomly and independently chosen from a disk $D(O,1)$. For simplicity, let $m=\frac{n}{5}$ be an integer. Distribute equidistantly $2m$ points $p_1,p_2,\dots,p_{2m}$ (in this order) on a circle C of radius $\frac{1}{2}$ centered at the point O. For every $i = 1, 2, \dots, 2m$, let R_i be the smaller of the two regions bounded by the boundary circle of the disk $D(O,1)$ and by the three lines $\overline{p_ip_{i+1}}$, $\overline{p_ip_{m+i+1}}$, and $\overline{p_{i+1}p_{m+i}}$ (see Fig. 3). The areas of the regions $a,b,\dots,g$ on the detailed Fig. 4 (which pictures a region R_i and "its" sector of the disk $D(O,1)$) satisfy $a+c+f=d$ and $e>b+g$. Consequently, $area(R_i)=d+e>\frac{a+b+c+d+e+f+g}{2}=\frac{\frac{1}{2m}\pi}{2}=\frac{1}{4m}\pi$. Thus, the probability that a region $R_i, i=1,2,\dots,2m$, contains a point of $\mathcal{P}$ is bigger than

$$1-\left(1-\frac{1}{4m}\right)^n.$$

For any $i=1,2,\dots,m$, the probability that both the regions R_i and R_{m+i} contain a point of $\mathcal{P}$ is bigger than

$$\left(1-\left(1-\frac{1}{4m}\right)^n\right)\left(1-\left(1-\frac{1}{4m}\right)^{n-1}\right).$$

It follows that the expected number of "occupied" pairs $(R_i,R_{m+i}), i=1,2,\dots,m$, is bigger than

$$m\left(1-\left(1-\frac{1}{4m}\right)^n\right)\left(1-\left(1-\frac{1}{4m}\right)^{n-1}\right)=$$
$$=\frac{n}{5}\left(1-\left(1-\frac{5}{4n}\right)^n\right)\left(1-\left(1-\frac{5}{4n}\right)^n-O\left(\frac{1}{n}\right)\right)\geq$$
$$\geq\frac{n}{5}\left(\left(1-e^{-\frac{5}{4}}\right)^2-O\left(\frac{1}{n}\right)\right)=\frac{\left(1-e^{-\frac{5}{4}}\right)^2}{5}n-O(1)>\frac{n}{10}-O(1).$$

By a standard probability argument, we can show that, with a high probability, there are at least $\frac{n}{10}-O(\sqrt{n})$ "occupied" pairs $(R_i,R_{m+i}), i=1,2,\dots,m$. Connecting a point of $R_i\cap\mathcal{P}$ with a point of $R_{m+i}\cap\mathcal{P}$ for every "occupied" pair (R_i,R_{m+i}), we obtain a crossing family with vertices in $\mathcal{P}$ of size at least $\frac{n}{10}-O(\sqrt{n})$.

The algorithm runs in time $O(n\log n)$.

(ii) Let B be any planar convex body of positive area. Ball [4] has shown that there exists an ellipsoid E inscribed to B whose area is at least a $\frac{\pi}{3\sqrt{3}}$-portion of the area of B. For any affine transformation T in the plane, it is the same

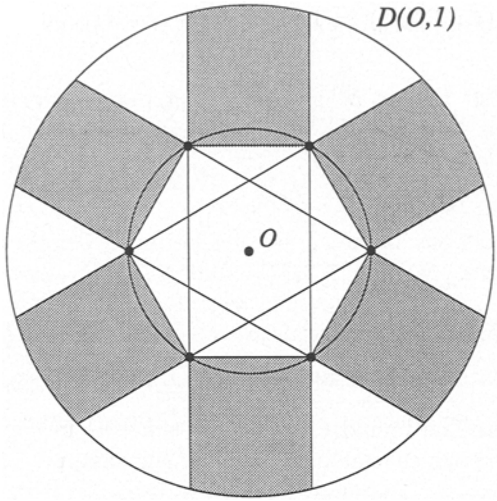

Fig. 3. Regions $R_i, i=1,2,\dots,2m$, in the case $m=3$.

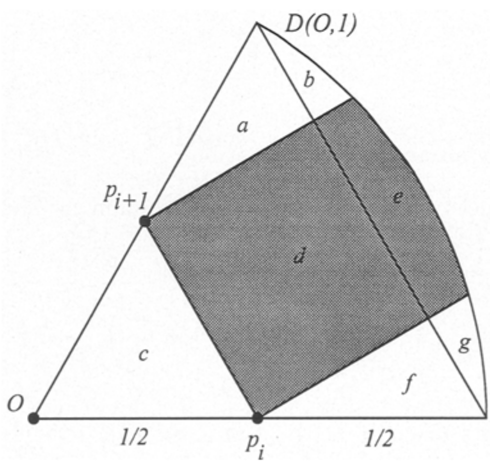

Fig. 4. Detail of the disk $D(O,1)$ with a region R_i

to place a point in B randomly with a uniform distribution and map it by T to $T(B)$ as to place a point in $T(B)$ randomly with a uniform distribution. Any affine transformation also preserves crossing families. Thus, we may assume without loss of generality that $E = D(O,1)$. In that case set $m = \left\lfloor \frac{\pi}{3\sqrt{3}} \cdot \frac{n}{5} \right\rfloor$ and define $2m$ regions $R_i, i=1,2,\dots,2m$, as in the proof of statement (i). The expected number of "occupied" pairs $(R_i, R_{m+i}), i=1,2,\ldots,m$, is now bigger than

$$m\left(1-\left(1-\frac{\pi}{3\sqrt{3}}\cdot\frac{1}{4m}\right)^{n}\right)\left(1-\left(1-\frac{\pi}{3\sqrt{3}}\cdot\frac{1}{4m}\right)^{n-1}\right)\geq$$

$$\geq\left\lfloor\frac{\pi}{3\sqrt{3}}\cdot\frac{n}{5}\right\rfloor\left(\left(1-e^{-\frac{5}{4}}\right)^{2}-O\left(\frac{1}{n}\right)\right)=\frac{\pi\cdot\left(\left(1-e^{-\frac{5}{4}}\right)^{2}\right.}{3\sqrt{3}\cdot 5}n-O(1)>$$

$$>\frac{6n}{100}-O(1).$$

Analogous arguments as in the above proof of statement (i) complete the proof. ∎

Concluding remarks. The bound $O(m^{2/3}n^{2/3}+m+n)$ in Theorem 1 is tight (up to a constant factor) which can be shown by an example of dense sets in case $\sqrt{n}\leq m\leq n^2$. This explains our interest in Conjecture 4. We conjecture that the above bound in Theorem 1 is tight only for "dense-like" sets:

Conjecture 30. *Let $\mathcal{P}$ be a set of n points in the plane and let $\mathcal{L}$ be a family of m lines in the plane. Suppose that $m=o(n^2)$, $n=o(m^2)$, and that the number of incidences between points in $\mathcal{P}$ and lines in $\mathcal{L}$ is at least $cm^{2/3}n^{2/3}$, for some constant $c>0$. Then there exist two constants $\alpha=\alpha(c)>\sqrt{2\sqrt{3}/\pi}$ and $c'=c'(c)>0$ such that the set $\mathcal{P}$ contains a subset of size at least $c'|\mathcal{P}|$ which can be transformed to an α-dense set by an affine transformation.*

Maybe the structure of sets achieving the bound in Theorem 1 is even more restrictive:

Conjecture 31. *Let the assumptions of Conjecture 31 be satisfied. Then there exists a constant $c'=c'(c)>0$ such that the set $\mathcal{P}$ contains a subset of size at least $c'|\mathcal{P}|$ which can be transformed to a subset of the square grid $\left\lfloor\sqrt{|\mathcal{P}|}\right\rfloor\times\left\lfloor\sqrt{|\mathcal{P}|}\right\rfloor$ by a projective transformation.*

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Pavel Valtr

Department of Applied Mathematics,
Charles University,
Malostranské nám. 25,
118 00 Praha 1, Czech Republic
valtr@kam.ms.mff.cuni.cz

and

Graduiertenkolleg
"Algorithmische Diskrete Mathematik",
Fachbereich Mathematik,
Freie Universität Berlin,
Takustrasse 9, 14195 Berlin,
Germany